Calculus 2 Rules

Made by mResource

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General rules

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Basic Integration

Variable of **Function**	**Integral**	**Conditions**
k (constant)	$\int k\,dx = kx + C$	
x^n	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	$n \neq -1$
$\frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\|x\| + C$	$x \neq 0$
e^x	$\int e^x\,dx = e^x + C$	
k^x	$\int k^x\,dx = \frac{k^x}{\ln k} + C$	$k > 0, k \neq 1$
$f(x)$	$\int \frac{f^\backslash(x)}{f(x)}\,dx = \ln\|f(x)\| + C$	
$\sqrt{f(x)}$	$\int \frac{f^\backslash(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C$	
$(f(x))^n$	$\int f^\backslash(x)(f(x))^n\,dx = \frac{(f(x))^{n+1}}{n+1} + C$	$f(x) \quad 1st \quad degree$

Trig Integration

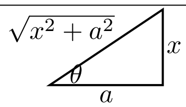
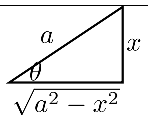
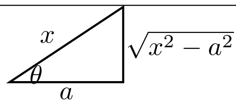
Trigonometric Functions **Square**	**Integral**
$\int \sin^2 u\,dx = \frac{u}{2} - \frac{sin(2u)}{4} + C$	$\int \sin x\,dx = -\cos x + C$
$\int \cos^2 u\,dx = \frac{u}{2} + \frac{sin(2u)}{4} + C$	$\int \cos x\,dx = \sin x + C$
$\int \tan^2 u\,dx = \tan u - u + C$	$\int \tan x\,dx = -\ln\|\cos x\| = \ln\|\sec x\| + C$
$\int \sec^2 u\,dx = \tan u + C$	$\int \sec x\,dx = \ln\|\tan x + \sec x\| + C$
$\int \csc^2 u\,dx = -\cot u + C$	$\int \csc x\,dx = \ln\|\csc x - \cot x\| + C$
$\int \cot^2 u\,dx = -\cot u - u + C$	$\int \cot x\,dx = \ln\|\sin x\| = -\ln\|\csc x\| + C$

General Trigonometric Rules:

The Rules:

$$1 + \tan^2 x = \sec^2 x \tag{1}$$
$$1 + \cot^2 x = \csc^2 x \tag{2}$$
$$\sin^2 x + \cos^2 x = 1 \tag{3}$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{4}$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \tag{5}$$
$$\sin 2x = 2 \sin x \cos x \tag{6}$$
$$\cos 2x = \cos^2 x - \sin^2 x \tag{7}$$
$$= \cos^2 x - 1 \tag{8}$$
$$= 1 - \sin^2 x \tag{9}$$
$$\tag{10}$$

Trigonometric Substitution Formulas

Expression	Substitution	Simplified Form	Diagram
$\sqrt{x^2 + a^2}$	$x = a \tan\theta$	$\sqrt{x^2 + a^2} = a \sec\theta$	
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$\sqrt{a^2 - x^2} = a \cos\theta$	
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$\sqrt{x^2 - a^2} = a \tan\theta$	

Trig Powers

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (\cos^2 x)^k \cos x \, dx$$
$$= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$
$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Strategy for Evaluating $\int \tan^n x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k, k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^n x \sec^{2k} x \, dx = \int \tan^n x (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^n x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k+1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

Then substitute $u = \sec x$.

$I_{n,m} = \int \sin^n x \ \cos^m x \, dx$

$I_{n,m} = \int \tan^n x \ \sec^m x \, dx$

***Note**

- $n \to m \to$ odd $\to$ Substitute the smallest
- n odd $\to$ m even $\to$ Substitute odd
- n even $\to$ m even $\to$ Substitute both of them

Sin & Cos Identities

$$\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)] \tag{11}$$

$$\cos a \sin b = \frac{1}{2}[\sin(a+b) - \sin(a-b)] \tag{12}$$

$$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)] \tag{13}$$

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)] \tag{14}$$

(15)

Partial Fractions

1. Denominator with Distinct Linear Factors

General Form

For denominators with distinct linear factors:

$$\frac{P(x)}{(x-a)(x-b)(x-c)\ldots} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)} + \ldots \tag{16}$$

2. Denominator with Repeated Linear Factors

General Form

For denominators with repeated linear factors:

$$\frac{P(x)}{(x-a)(x-b)^2(x-c)^3\dots} = \frac{A}{(x-a)} + \frac{B_1}{(x-b)} + \frac{B_2}{(x-b)^2} + \frac{C_1}{(x-c)} + \frac{C_2}{(x-c)^2} + \frac{C_3}{(x-c)^3} + \dots \tag{17}$$

3. Denominator with Quadratic Factors

General Form

For denominators with irreducible quadratic factors:

$$\frac{P(x)}{x(x^2+ax+b)} = \frac{A}{x} + \frac{Bx+C}{x^2+ax+b} \tag{18}$$

For repeated quadratic factors:

$$\frac{P(x)}{x(x^2+ax+b)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+ax+b} + \frac{Dx+E}{(x^2+ax+b)^2} \tag{19}$$

4. Denominator with Second-Degree Equations

General Form

For denominators with second-degree equations:

$$\frac{P(x)}{(x^2-a)^2(x^2-b)^2\dots} = \frac{Ax+B}{(x^2-a)} + \frac{Cx+D}{(x^2-a)^2} + \frac{A_2x+B_2}{(x^2-b)} + \frac{C_2x+D_2}{(x^2-b)^2} + \dots \tag{20}$$

Completing Square

Main Formula

$$\int \frac{Ax+B}{x^2+bx+c}\,dx \tag{21}$$

$$OR \int \frac{Ax+B}{\sqrt{x^2+bx+c}}\,dx \tag{22}$$

(23)

General Form

$$x^2+bx+c=0 \tag{24}$$

$$(x+\frac{b}{2})^2-(\frac{b}{2})^2+c=0 \tag{25}$$

(26)

Notes:

- $\sin^{-1}\mathbf{x} \rightarrow \frac{1}{\sqrt{1-x^2}}$
- $\cos^{-1}\mathbf{x} \rightarrow \frac{-1}{\sqrt{1-x^2}}$
- $\tan^{-1}\mathbf{x} \rightarrow \frac{1}{1+x^2}$
- $\cot^{-1}\mathbf{x} \rightarrow \frac{-1}{1+x^2}$
- $\sec^{-1}\mathbf{x} \rightarrow \frac{1}{x\sqrt{x^2-1}}$
- $\csc^{-1}\mathbf{x} \rightarrow \frac{-1}{x\sqrt{x^2-1}}$
- $\sinh^{-1}\mathbf{x} \rightarrow \frac{1}{\sqrt{x^2+1}}$
- $\cosh^{-1}\mathbf{x} \rightarrow \frac{1}{\sqrt{x^2-1}}$
- $\tanh^{-1}\mathbf{x} = \coth^{-1}\mathbf{x} \rightarrow \frac{1}{1-x^2}$
- $\text{sech}^{-1}\mathbf{x} \rightarrow \frac{-1}{x\sqrt{1-x^2}}$
- $\text{csch}^{-1}\mathbf{x} \rightarrow \frac{-1}{|x|\sqrt{1+x^2}}$

Arc Length

Arc Length Formula

For curves of the form $y = f(x)$

If f is continuous and differentiable on the interval $[a, b]$, then the arc length is:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

For curves of the form $x = g(y)$

If g is continuous and differentiable on the interval $[c, d]$, then the arc length is:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Examples

Example 1. *Find the arc length of $y = \frac{2}{3}x^{3/2}$ from $x = 1$ to $x = 2$.*

Solution:

$$f(x) = \frac{2}{3}x^{3/2}, \quad f'(x) = \sqrt{x}$$

$$L = \int_1^2 \sqrt{1+x}\, dx = \int_2^3 \sqrt{u}\, du \quad \text{where } u = 1 + x$$

$$= \frac{2}{3}\left(3^{3/2} - 2^{3/2}\right) \simeq 1.578.$$

Example 2. *Find the arc length of $y = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = 2$.*

Solution:

$$f(x) = \frac{1}{2}(e^x + e^{-x}), \quad f'(x) = \frac{1}{2}(e^x - e^{-x})$$

$$[f'(x)]^2 = \frac{1}{4}(e^x - e^{-x})^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$L = \int_0^2 \sqrt{1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})}\, dx = \int_0^2 \frac{1}{2}(e^x + e^{-x})\, dx$$

$$= \frac{1}{2}(e^2 - e^{-2}) \simeq 1.291.$$

Reduction formulas

$$I_n = \int \sin^n x\, dx = \int \sin^{n-1} x\, \sin x\, dx$$

Using integration by parts:

$$u = \sin^{n-1} x \Rightarrow dv = \sin x$$
$$du = (n-1)\sin^{n-2} x \cos x\, dx \Rightarrow V = -\cos x$$

$$\begin{aligned}
I_n &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, \cos^2 x\, dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\, dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx \\
&= -\cos x \sin^{n-1} x + (n-1)\,[I_{n-2} - (n-1)I_n] \\
I_n + (n-1)I_n &= -\cos x \sin^{n-1} x + (n-1)I_{n-2} \\
nI_n &= -\cos x \sin^{n-1} x + (n-1)I_{n-2} \\
I_n &= \frac{-1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}
\end{aligned}$$

2. $I_n = \int \cos^n x\, dx = \int \cos^{n-1} x\, \cos x\, dx$

Using integration by parts:

$$u = \cos^{n-1} x \Rightarrow dv = \cos x$$
$$du = (n-1)\cos^{n-2} x(-\sin x)\, dx \Rightarrow V = \sin x$$

$$\begin{aligned}
I_n &= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\, \sin^2 x\, dx \\
&= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)\, dx \\
&= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx \\
&= \sin x \cos^{n-1} x + (n-1)\,[I_{n-2} - (n-1)I_n] \\
I_n + (n-1)I_n &= \sin x \cos^{n-1} x + (n-1)I_{n-2} \\
nI_n &= \sin x \cos^{n-1} x + (n-1)I_{n-2} \\
I_n &= \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}
\end{aligned}$$

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3. $I_n = \int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx$

Using integration by parts:

$$\begin{aligned}
u &= \sec^{n-2} x \\
dv &= \sec^2 x\, dx \\
du &= (n-2)\sec^{n-3} x \sec x \tan x\, dx \\
&= (n-2)\sec^{n-2} x \tan x\, dx \\
v &= \tan x
\end{aligned}$$

$$
\begin{aligned}
I_n &= \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx \\
&= \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx \\
&= \tan x \sec^{n-2} x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx \\
&= \tan x \sec^{n-2} x - (n-2) I_n + (n-2) I_{n-2}
\end{aligned}
$$

$$
\begin{aligned}
I_n + (n-2) I_n &= \tan x \sec^{n-2} x + (n-2) I_{n-2} \\
(n-1) I_n &= \tan x \sec^{n-2} x + (n-2) I_{n-2} \\
I_n &= \frac{1}{n-1} \tan x \sec^{n-2} x + \frac{n-2}{n-1} I_{n-2}
\end{aligned}
$$

4. $I_n = \int \csc^n x\, dx = \int \csc^{n-2} x \csc^2 x\, dx$

Using integration by parts:

$$\begin{aligned} u &= \csc^{n-2} x \\ dv &= \csc^2 x\, dx \\ du &= (n-2)\csc^{n-3} x(-\csc x \cot x)\, dx \\ &= -(n-2)\csc^{n-2} x \cot x\, dx \\ v &= -\cot x \end{aligned}$$

$$\begin{aligned} I_n &= -\cot x \csc^{n-2} x - (n-2)\int \csc^{n-2} x \cot^2 x\, dx \\ &= -\cot x \csc^{n-2} x - (n-2)\int \csc^{n-2} x(\csc^2 x - 1)\, dx \\ &= -\cot x \csc^{n-2} x - (n-2)\int \csc^n x\, dx + (n-2)\int \csc^{n-2} x\, dx \\ &= -\cot x \csc^{n-2} x - (n-2)I_n + (n-2)I_{n-2} \end{aligned}$$

$$\begin{aligned} I_n + (n-2)I_n &= -\cot x \csc^{n-2} x + (n-2)I_{n-2} \\ (n-1)I_n &= -\cot x \csc^{n-2} x + (n-2)I_{n-2} \\ I_n &= \frac{-1}{n-1}\cot x \csc^{n-2} x + \frac{n-2}{n-1}I_{n-2} \end{aligned}$$

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5. $I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \tan^2 x\, dx$

$$I_n = \int \tan^{n-2} x(\sec^2 x - 1)\, dx \tag{27}$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2}\; dx \tag{28}$$

$$I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-1} \tag{29}$$

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6. $I_n = \int \cot^n x\, dx = \int \cot^{n-2} x \cot^2 x\, dx$

$$I_n = \int \cot^{n-2} x(\csc^2 x - 1)\, dx \tag{30}$$

$$= \int \cot^{n-2} x \csc^2 x\, dx - \int \cot^{n-2}\; dx \tag{31}$$

$$I_n = \frac{1}{n-1}\cot^{n-1} x - I_{n-1} \tag{32}$$

7. $I_n = \int (\ln x)^n \, dx$

$$I_n = \int (\ln x)^n dx$$

$$\text{Let } u = (\ln x)^n \quad du = n(\ln x)^{n-1} \cdot \frac{1}{x} dx$$

$$dv = dx \quad v = x$$

$$I_n = x(\ln x)^n - \int x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x} dx$$

$$= x(\ln x)^n - n \int (\ln x)^{n-1} dx$$

$$I_n = x(\ln x)^n - nI_{n-1}$$