Discrete Structures

Summary

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Discrete Mathematics

1. Logic and Proofs

Logic forms the foundation of all mathematical reasoning and is essential for understanding discrete structures.

1.1 Propositional Logic

Definition 1.1 — Proposition. A proposition is a declarative statement that is either true or false, but not both.

Example 1.1 — Propositions.
- "The sky is blue." (True)
- "2 + 2 = 5." (False)
- "What time is it?" (Not a proposition - it's a question)
- "Close the door." (Not a proposition - it's a command)

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1.2 Logical Connectives

The basic logical connectives are:
- **Negation** ($\neg p$): "not p"
- **Conjunction** ($p \wedge q$): "p and q"
- **Disjunction** ($p \vee q$): "p or q" (inclusive OR)
- **Exclusive OR** ($p \oplus q$): "p or q but not both" (exclusive OR)
- **Implication** ($p \rightarrow q$): "if p then q"
- **Biconditional** ($p \leftrightarrow q$): "p if and only if q"

R **Inclusive vs. Exclusive OR:** The standard disjunction ($\vee$) is inclusive, meaning "p or q or both." The exclusive OR ($\oplus$) means "p or q but not both."

1.3 Conditional Statements

For a conditional statement $p \rightarrow q$:
- p is called the **hypothesis** or **antecedent**
- q is called the **conclusion** or **consequent**

1.3.1 Related Conditional Statements

Given the conditional $p \rightarrow q$:
- **Converse**: $q \rightarrow p$

- **Contrapositive**: $\neg q \rightarrow \neg p$
- **Inverse**: $\neg p \rightarrow \neg q$

Theorem 1.1 — Equivalence of Conditional and Contrapositive. A conditional statement and its contrapositive are logically equivalent:

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

1.4 Truth Tables for Compound Propositions

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Table 1.1: Truth tables for basic logical connectives

1.5 Precedence of Logical Connectives

The order of precedence (from highest to lowest):

- Negation ($\neg$)
- Conjunction ($\wedge$) and Disjunction ($\vee$)
- Implication ($\rightarrow$)
- Biconditional ($\leftrightarrow$)

1.6 Logical Equivalence

Definition 1.2 — Logical Equivalence. Two propositions are logically equivalent if they have the same truth value for all possible truth assignments. We denote this as $p \equiv q$ or $p \Leftrightarrow q$.

Logical Equivalences:

Logical Equivalences Involving Conditional Statements:

Logical Equivalences Involving Biconditional Statements:

1.7 Logic and Bit Operations

Logical operations correspond to bitwise operations on computer systems:

- AND ($\wedge$) corresponds to bitwise AND
- OR ($\vee$) corresponds to bitwise OR
- XOR ($\oplus$) corresponds to bitwise XOR
- NOT ($\neg$) corresponds to bitwise complement

1.8 Translation from English to Propositional Logic

■ **Example 1.2 — English to Logic Translation.** Let p: "It is raining" and q: "I will go to the store"

- "If it is raining, then I will not go to the store" $\rightarrow p \rightarrow \neg q$
- "I will go to the store unless it is raining" $\rightarrow \neg p \rightarrow q$
- "It is raining and I will go to the store" $\rightarrow p \wedge q$

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Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

1.9 Tautologies, Contradictions, and Contingencies

Definition 1.3 — Tautology. A tautology is a compound proposition that is always true, regardless of the truth values of its component propositions.

Definition 1.4 — Contradiction. A contradiction is a compound proposition that is always false, regardless of the truth values of its component propositions.

Definition 1.5 — Contingency. A contingency is a compound proposition that is neither a tautology nor a contradiction.

1.10 De Morgan's Laws

Theorem 1.2 — De Morgan's Laws. For any propositions p and q:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \tag{1.1}$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \tag{1.2}$$

Logical Equivalences Involving Conditional Statements
$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

Equivalences Involving Biconditional Statements
$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

1.11 Predicate Logic

Definition 1.6 — Predicate. A predicate is a statement involving variables that becomes a proposition when the variables are assigned specific values from a domain.

1.11.1 Quantifiers

- **Universal Quantifier** ($\forall x$): "for all x"
- **Existential Quantifier** ($\exists x$): "there exists an x"

1.12 Arguments and Validity

Definition 1.7 — Argument. An argument is a sequence of propositions. All but the final proposition are called premises, and the final proposition is called the conclusion.

Definition 1.8 — Valid Argument. An argument is valid if the conclusion is true whenever all the premises are true.

1.13 Rules of Inference

Rule of Inference	Tautology	Name
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$(p \land q) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Table 1.2: Rules of inference

1.14 Rules of Inference for Quantified Statements

Rule of Inference	Name
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Table 1.3: Rules of inference for quantified statements

1.15 Methods of Proof

1.15.1 Direct Proof

In a direct proof of $p \to q$:

- Assume that p is true
- Use rules of inference and logical equivalences to show that q is true
- Conclude that $p \to q$ is true

Definition 1.9 — Common Number Types for Proofs. Common number types used in mathematical proofs:

- **Even integer**: $n = 2k$ for some integer k
- **Odd integer**: $n = 2k+1$ for some integer k
- **Perfect square**: $n = k^2$ for some integer k
- **Rational number**: $r = \frac{m}{n}$ where m, n are integers with no common factors and $n \neq 0$

1.15.2 Indirect Proof

1.15.2.1 Proof by Contrapositive

We use the logical equivalence: $p \to q \equiv \neg q \to \neg p$

Steps:

- Assume $\neg q$ is true
- Show that $\neg p$ is true
- Conclude $\neg q \to \neg p$ is true
- Therefore, $p \to q$ is true by logical equivalence

■ **Example 1.3 — Proof by Contrapositive. Prove:** If n^2 is even, then n is even.

Contrapositive: If n is odd, then n^2 is odd.

- Assume n is odd: $n = 2k+1$
- Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd
- Thus, the contrapositive is true, so the original statement is true

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1.15.2.2 Proof by Contradiction

To prove a statement P is true, assume the opposite, $\neg P$, and show that it leads to a contradiction.

Steps for One Statement:

- Assume $\neg P$ is true
- Show this leads to a contradiction
- Therefore, $\neg P$ is false, so P is true

Steps for Implication Statement $P \to Q$:

- Assume P is true and $\neg Q$ is true (i.e., $P \wedge \neg Q$)
- Show that this leads to a contradiction
- Therefore, $P \wedge \neg Q$ is false, so $P \to Q$ is true

■ **Example 1.4 — Proof by Contradiction. Prove:** $\sqrt{2}$ is irrational

- Assume the opposite: $\sqrt{2}$ is rational
- Then $\sqrt{2} = \frac{a}{b}$ where a, b are coprime integers
- $2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2$
- So a^2 is even $\Rightarrow$ a is even $\Rightarrow$ $a = 2k$
- Then $a^2 = 4k^2 \Rightarrow 2b^2 = 4k^2 \Rightarrow b^2 = 2k^2 \Rightarrow b$ is even
- Contradiction: both a and b are even, so not coprime
- Therefore, $\sqrt{2}$ is irrational

■

2. Counting and Combinatorics

Combinatorics is the study of counting discrete objects and is fundamental to probability theory and algorithm analysis.

2.1 Basic Counting Principles

2.1.1 The Sum Rule

Definition 2.1 — Sum Rule. If a task can be performed in m ways, while another task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in $m+n$ ways.

Set theoretical version of the sum rule: If A and B are disjoint sets ($A \cap B = \emptyset$) then

$$|A \cup B| = |A| + |B|$$

More generally, if the sets $A_1, A_2, \ldots, A_n$ are pairwise disjoint, then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

■ **Example 2.1 — Sum Rule Application.** If a class has 30 male students and 25 female students, then the class has $30+25=55$ students. ■

2.1.2 The Product Rule

Definition 2.2 — Product Rule. If a task can be performed in m ways and another independent task can be performed in n ways, then the combination of both tasks can be performed in mn ways.

Set theoretical version of the product rule: Let $A \times B$ be the Cartesian product of sets A and B. Then:

$$|A \times B| = |A| \cdot |B|$$

More generally:

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$$

■ **Example 2.2 — License Plates.** Assume that a license plate contains two letters followed by three digits. Each letter can be printed in 26 ways, and each digit can be printed in 10 ways, so $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 676{,}000$ different plates can be printed. ■

2.1.3 The Inclusion-Exclusion Principle

Definition 2.3 — Inclusion-Exclusion Principle. The inclusion-exclusion principle generalizes the sum rule to non-disjoint sets.

In general, for arbitrary (but finite) sets A, B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

■ **Example 2.3 — University Students.** Assume that in a university with 1000 students, 200 students are taking a course in mathematics, 300 are taking a course in physics, and 50 students are taking both. How many students are taking at least one of those courses?

Answer: If M = set of students taking Mathematics, P = set of students taking Physics, then:

$$|M \cup P| = |M| + |P| - |M \cap P| = 200 + 300 - 50 = 450$$

students are taking Mathematics or Physics. ■

For three sets the following formula applies:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

And for an arbitrary union of sets:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = s_1 - s_2 + s_3 - s_4 + \cdots \pm s_n$$

where s_k = sum of the cardinalities of all possible k-fold intersections of the given sets.

3. Functions

Functions are fundamental mathematical objects that describe relationships between sets and form the basis for understanding mappings in discrete mathematics.

3.1 Definition and Basic Concepts

Definition 3.1 — Function. Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Definition 3.2 — Domain, Codomain, Range. If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f. If $f(a) = b$, we say that b is the **image** of a and a is a **preimage** of b. The **range**, or **image**, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f **maps** A to B.

■ **Example 3.1 — Function Example.** Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$. Define a function $f : A \to B$ by:

$$f(1) = a, \quad f(2) = b, \quad f(3) = d$$

Each element of A is assigned exactly one element of B, so f is a valid function from A to B. ■

3.2 Types of Functions

3.2.1 One-to-One Functions (Injective)

Definition 3.3 — Injective Function. A function f is said to be **one-to-one**, or an **injection**, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be **injective** if it is one-to-one.

R A function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

We can express that f is one-to-one using quantifiers as

$$\forall a \forall b\,(f(a) = f(b) \to a = b)$$

or equivalently,

$$\forall a \forall b\,(a \neq b \to f(a) \neq f(b))$$

where the universe of discourse is the domain of the function.

■ **Example 3.2 — Non-Injective Function.** Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution: The function $f(x) = x^2$ is **not** one-to-one because, for instance,

$$f(1) = f(-1) = 1$$

but $1 \neq -1$. Therefore, different inputs can give the same output, so the function is not injective.

Note that the function $f(x) = x^2$, with its domain restricted to $\mathbb{Z}^+$ (the set of positive integers), **is** one-to-one. ■

3.2.2 Onto Functions (Surjective)

Definition 3.4 — Surjective Function. A function f from A to B is called **onto**, or a **surjection**, if and only if for every element $b \in B$, there is an element $a \in A$ such that $f(a) = b$. A function f is called **surjective** if it is onto.

R A function f is **onto** if

$$\forall y \exists x (f(x) = y)$$

where the domain for x is the domain of the function and the domain for y is the codomain of the function.

■ **Example 3.3 — Non-Surjective Function.** Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: The function f is **not** onto because there is no integer x such that

$$x^2 = -1$$

for instance. Therefore, not every element in the codomain $\mathbb{Z}$ has a preimage in the domain $\mathbb{Z}$. ■

3.2.3 Bijective Functions

Definition 3.5 — Bijective Function. The function f is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto. We also say that such a function is **bijective**.

■ **Example 3.4 — Bijection Example.** Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with

$$f(a) = 4, \quad f(b) = 2, \quad f(c) = 1, \quad f(d) = 3$$

Is f a bijection?

Solution: The function f is **one-to-one** and **onto**.

It is one-to-one because no two values in the domain are assigned the same function value.

It is onto because all four elements of the codomain $\{1, 2, 3, 4\}$ are images of elements in the domain $\{a, b, c, d\}$.

Hence, f is a **bijection**. ■

3.3 Inverse Functions and Composition

Definition 3.6 — Inverse Function. Let f be a one-to-one correspondence from the set A to the set B. The **inverse function** of f is the function that assigns to each element $b \in B$ the unique element $a \in A$ such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

R Be sure not to confuse the function f^{-1} with the function $\frac{1}{f}$, which is the function that assigns to each x in the domain the value $\frac{1}{f(x)}$. Notice that the latter makes sense only when $f(x)$ is a non-zero real number.

■ **Example 3.5 — Inverse Function Example.** Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that

$$f(a) = 2, \quad f(b) = 3, \quad f(c) = 1$$

Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence (bijection). The inverse function f^{-1} reverses the correspondence given by f, so

$$f^{-1}(1) = c, \quad f^{-1}(2) = a, \quad f^{-1}(3) = b$$

■

3.3.1 Function Composition

Definition 3.7 — Function Composition. Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The **composition** of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.

In other words, $f \circ g$ is the function that assigns to the element a of A the element assigned by f to $g(a)$. That is, to find $(f \circ g)(a)$, we first apply the function g to a to obtain $g(a)$ and then we apply the function f to the result $g(a)$ to obtain $(f \circ g)(a) = f(g(a))$.

Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

■ **Example 3.6 — Function Composition.** Let f and g be functions from the set of integers to itself, defined by:

$$f(x) = 2x+3 \quad \text{and} \quad g(x) = 3x+2$$

What is the composition of f and g? What is the composition of g and f?

Solution: Both compositions $f \circ g$ and $g \circ f$ are defined. We compute:

$$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)+3 = 6x+7$$

$$(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3)+2 = 6x+11$$

Remark: Even though $f \circ g$ and $g \circ f$ are both defined, they are not equal. In other words, **composition of functions is not generally commutative**. ■

When composing a function with its inverse, in either order, we obtain the identity function. Suppose f is a one-to-one correspondence from a set A to a set B. Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A. Since $f^{-1}(b) = a$ when $f(a) = b$, we have:

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$$

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$$

Thus, $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$, where I_A and I_B are the identity functions on sets A and B, respectively. That is,

$$(f^{-1})^{-1} = f$$

3.4 Special Functions

3.4.1 Floor and Ceiling Functions

Definition 3.8 — Floor and Ceiling Functions. The **floor function** assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$.

The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

R The floor function is often also called the greatest integer function. It is often denoted by $\lfloor x \rfloor$ or $[x]$.

■ **Example 3.7 — Floor and Ceiling Examples.** These are some values of the floor and ceiling functions:

$$\lfloor 1/2 \rfloor = 0, \quad \lceil 1/2 \rceil = 1, \quad \lfloor -1/2 \rfloor = -1, \quad \lceil -1/2 \rceil = 0$$
$$\lfloor 3.1 \rfloor = 3, \quad \lceil 3.1 \rceil = 4, \quad \lfloor 7 \rfloor = 7, \quad \lceil 7 \rceil = 7$$

■

4. Sequences and Summations

Sequences and summations are fundamental concepts in discrete mathematics that provide tools for describing patterns and calculating sums efficiently.

4.1 Sequences

Definition 4.1 — Sequence. A **sequence** is a function from a subset of the set of integers (usually either the set $\{0,1,2,\dots\}$ or the set $\{1,2,3,\dots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a *term* of the sequence.

▪ **Example 4.1 — Sequence Example.** Consider the sequence $\{a_n\}$, where $a_n = \frac{1}{n}$. The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \dots$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

■

4.1.1 Finite and Infinite Sequences

A sequence is called **finite** if it contains a specific number of terms and ends, while it is **infinite** if it continues indefinitely without terminating.

4.1.2 Types of Sequences

4.1.2.1 Arithmetic Sequences

Definition 4.2 — Arithmetic Progression. An arithmetic progression is a sequence of the form

$$a, a+d, a+2d, a+3d, \dots, a+nd, \dots$$

where the initial term a and the common difference d are real numbers.

The n-th term of an arithmetic sequence is given by the formula:

$$a_n = a + (n-1) \cdot d$$

where:

- a_n is the n-th term

- a is the first term
- d is the common difference
- n is the position of the term in the sequence

R An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

Definition 4.3 — Common Difference. In an arithmetic sequence, the *common difference* d is the fixed amount added to each term to get the next term. It can be found by subtracting any term from the term that follows it:

$$d = a_{n+1} - a_n$$

■ **Example 4.2 — Arithmetic Sequence.** Consider the arithmetic sequence with first term $a = 2$ and common difference $d = 3$.

$$a_n = a + (n-1)d = 2 + (n-1) \cdot 3$$

The first few terms are:

$$2, 5, 8, 11, 14, \ldots$$

■

4.1.2.2 Geometric Sequences

Definition 4.4 — Geometric Progression. A geometric progression is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the initial term a and the common ratio r are real numbers.

The n-th term of a geometric sequence is given by:

$$a_n = a \cdot r^{n-1}$$

where:

- a_n is the n-th term
- a is the first term
- r is the common ratio
- n is the position of the term in the sequence

R A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

Definition 4.5 — Common Ratio. It is the fixed number by which each term is multiplied to get the next term. It can be found using the formula:

$$r = \frac{a_{n+1}}{a_n}$$

■ **Example 4.3 — Geometric Sequence.** Consider a geometric sequence with first term $a = 3$ and common ratio $r = 2$.

$$a_n = a \cdot r^{n-1} = 3 \cdot 2^{n-1}$$

The first few terms of the sequence are:

$$3, 6, 12, 24, 48, \ldots$$

■

4.1.2.3 Special Sequences

- **Fibonacci Sequence**: Each term is the sum of the two preceding terms

$$a_n = a_{n-1} + a_{n-2}, \quad a_1 = 1, a_2 = 1$$

Example: $1, 1, 2, 3, 5, 8, 13, \ldots$
- **Harmonic Sequence**: Reciprocals of positive integers

$$a_n = \frac{1}{n}$$

Example: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
- **Factorial Sequence**: Each term is the factorial of its position

$$a_n = n!$$

Example: $1, 2, 6, 24, 120, \ldots$
- **Square Numbers**: Squares of natural numbers

$$a_n = n^2$$

Example: $1, 4, 9, 16, 25, \ldots$
- **Triangular Numbers**: Sum of the first n natural numbers

$$a_n = \frac{n(n+1)}{2}$$

Example: $1, 3, 6, 10, 15, \ldots$

4.2 Summations

Definition 4.6 — Summation. The summation of a sequence of terms is the process of adding them together. It is typically represented as:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

- $\sum$ denotes the summation symbol
- a_i represents the terms being summed
- i is the index of summation

4.2.1 Examples of Summations

Example 4.4 — Simple Summation. What is the value of $\sum_{i=1}^{4} 2i$?
Solution:

$$\sum_{i=1}^{4} 2i = 2(1) + 2(2) + 2(3) + 2(4) = 2 + 4 + 6 + 8 = 20$$

■

Example 4.5 — Alternating Summation. What is the value of $\sum_{k=4}^{8} (-1)^k$?
Solution: We have

$$\sum_{k=4}^{8} (-1)^k = (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$$

$$= 1 + (-1) + 1 + (-1) + 1 = 1$$

■

4.2.2 Common Summation Formulas

Theorem 4.1 — Important Summation Formulas.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{4.1}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{4.2}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{4.3}$$

$$\sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r} \quad (r \neq 1) \tag{4.4}$$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad (|r| < 1) \tag{4.5}$$

4.2.3 Properties of Summations

Let a_k, b_k be sequences and c be a constant. Then:

1. **Linearity:**

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

2. **Constant Multiple:**

$$\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$$

3. **Summation of a Constant:**

$$\sum_{k=1}^{n} c = c \cdot n$$

4. **Index Shifting:**

$$\sum_{k=m}^{n} a_k = \sum_{j=0}^{n-m} a_{j+m} \quad (\text{where } j = k - m)$$

5. **Splitting a Sum:**

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k \quad \text{for } 1 \leq m < n$$

4.2.4 Sum of a Geometric Sequence

A geometric sequence is a sequence where each term is found by multiplying the previous term by a constant ratio r.

4.2.4.1 Finite Geometric Series

If a is the first term and $r \neq 1$, the sum of the first n terms is:

$$S_n = a\frac{1-r^n}{1-r}$$

4.2.4.2 Infinite Geometric Series

If $|r| < 1$, the sum of an infinite geometric series is:

$$S = \frac{a}{1-r}$$

■ **Example 4.6 — Finite Geometric Series.** Given the geometric sequence: $3, 6, 12, 24, 48$, we have:

- First term: $a = 3$
- Common ratio: $r = 2$
- Number of terms: $n = 5$

Using the formula:

$$S_5 = 3 \cdot \frac{1-2^5}{1-2} = 3 \cdot \frac{1-32}{-1} = 3 \cdot 31 = 93$$

■

5. Graph Theory

Graph theory provides powerful tools for modeling and solving real-world problems involving relationships between objects.

5.1 Introduction to Graphs

5.1.1 Definition of a Graph

A **graph** $G = (V, E)$ is a mathematical structure consisting of:

- A set V of points called **vertices** (singular: vertex)
- A set E of lines called **edges** that connect some of the vertices

5.1.2 Applications

Graphs can visually depict links between objects where:

- Objects are represented as **vertices**
- Links between objects are represented as **edges**

5.1.3 Graph Properties

- Edges can overlap with other edges
- There does not need to be a vertex at edge overlaps
- Graphs can consist of multiple disconnected pieces
- Vertices can exist independently with no incident edges

5.2 Graph Notation and Terminology

5.2.1 Basic Notation

Let $G = (V, E)$ be a graph. We define:

$$v = |V| \quad \text{(number of vertices, also called the \textbf{order})} \tag{5.1}$$

$$e = |E| \quad \text{(number of edges)} \tag{5.2}$$

5.2.2 Vertex Sets

The **vertex set** of a graph is denoted as:

$$V = \{v_1, v_2, v_3, \ldots, v_n\}$$

For a graph with labeled vertices A, B, C, D, we write:

$$V = \{A, B, C, D\}$$

5.2.3 Edge Notation

An edge connecting vertices u and v is denoted as:

$$e = \{u, v\} \quad \text{or} \quad e = (u, v)$$

The complete edge set can be written as:

$$E = \{e_1, e_2, e_3, \ldots, e_m\}$$

5.3 Vertex Degrees

5.3.1 Definition

The **degree** of a vertex v, denoted $\deg(v)$ or $d(v)$, is the number of edges incident to (attached to) that vertex.

5.3.2 Degree List

The **degree list** of a graph is a sequence of the degrees of all vertices, arranged in non-decreasing order:

$$\text{Degree List} = [d_1, d_2, d_3, \ldots, d_n]$$

where $d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_n$.

5.3.3 Handshaking Lemma

Theorem 5.1 — Handshaking Lemma. For any graph $G = (V, E)$:

$$\sum_{v \in V} \deg(v) = 2|E|$$

This means the sum of all vertex degrees equals twice the number of edges.

5.4 Planar Graphs

5.4.1 Definition

A graph is **planar** if it can be drawn in the plane such that no two edges cross each other.

5.4.2 Faces of a Planar Graph

When a connected planar graph is drawn without edge crossings, it divides the plane into regions called **faces**. This includes:

- **Interior faces**: Bounded regions
- **Exterior face**: The unbounded region surrounding the graph

5.4.3 Degree of a Face

The **degree of a face** is the number of edges that bound that face. An edge that bounds only one face (like a bridge) contributes 1 to the face's degree, while an edge bounding two faces contributes 1 to each face's degree.

5.5 Euler's Formula

5.5.1 Euler's Formula for Planar Graphs

Theorem 5.2 — Euler's Formula. For a connected planar graph with v vertices, e edges, and f faces:

$$v - e + f = 2$$

This is one of the most fundamental results in graph theory.

■ **Example 5.1 — Euler's Formula Application. Problem:** A connected planar graph has 24 vertices and 30 faces. How many edges does the graph have?

Solution: Using Euler's formula:

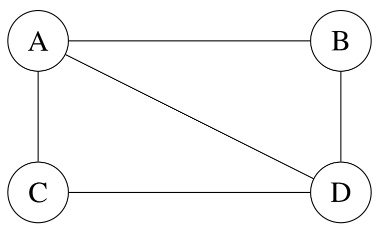
$$v - e + f = 2 \quad (5.3)$$
$$24 - e + 30 = 2 \quad (5.4)$$
$$54 - e = 2 \quad (5.5)$$
$$e = 52 \quad (5.6)$$

Therefore, the graph has 52 edges. ■

5.6 Graph Examples

5.6.1 Simple Graph Example

Consider a graph with 4 vertices labeled A, B, C, and D, with 5 edges connecting them:

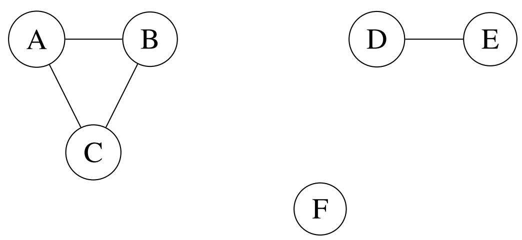


Analysis:

- Vertex set: $V = \{A, B, C, D\}$
- Edge set: $E = \{\{A,B\},\{A,C\},\{A,D\},\{B,D\},\{C,D\}\}$
- Number of vertices: $v = 4$
- Number of edges: $e = 5$
- Degrees: $\deg(A) = 3$, $\deg(B) = 2$, $\deg(C) = 2$, $\deg(D) = 3$
- Degree list: $[2, 2, 3, 3]$

5.6.2 Disconnected Graph Example

Here's an example of a graph with multiple components:

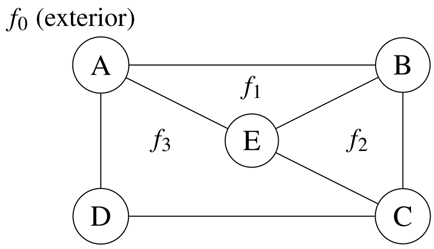


Analysis for a Multi-Component Graph:

- A graph with 6 vertices and 4 edges
- It consists of 3 connected components:
  1. Triangle: vertices $\{A,B,C\}$ with 3 edges
  2. Edge: vertices $\{D,E\}$ with 1 edge
  3. Isolated vertex: $\{F\}$ with 0 edges
- Degrees: $\deg(A) = 2$, $\deg(B) = 2$, $\deg(C) = 2$, $\deg(D) = 1$, $\deg(E) = 1$, $\deg(F) = 0$
- Degree list: $[0,1,1,2,2,2]$

5.6.3 Planar Graph with Faces

Consider this planar graph and its faces:



Euler's Formula Verification:
- Vertices: $v = 5$
- Edges: $e = 7$
- Faces: $f = 4$ (including exterior face f_0)
- Check: $v - e + f = 5 - 7 + 4 = 2$

Face Degrees:
- $\deg(f_0) = 4$ (exterior face)
- $\deg(f_1) = 3$ (interior face)
- $\deg(f_2) = 3$ (interior face)
- $\deg(f_3) = 4$ (interior face)

5.7 Graph Theory Applications

Graph theory has numerous applications across various fields:
- **Computer Science**: Network topology, data structures, algorithms
- **Social Networks**: Modeling relationships and connections
- **Transportation**: Route optimization, traffic flow analysis
- **Biology**: Protein interactions, evolutionary trees
- **Economics**: Market analysis, supply chain optimization
- **Engineering**: Circuit design, structural analysis

Understanding fundamental concepts like vertex degrees, planarity, and Euler's formula forms the foundation for more advanced graph theoretical applications in computer science, mathematics, and engineering.

5.8 Related Mathematical Concepts

5.8.1 Fuzzy Logic

Instead of things being just "true" or "false," fuzzy logic lets things be partially true. For example, someone can be "somewhat tall" or "very warm." This is more like how humans think in real life.

It's used in everyday items like air conditioners and washing machines to make smart decisions when things aren't black and white.

5.8.2 Graphs and Trees

A graph is just dots (called nodes) connected by lines (called edges). Think of a map with cities connected by roads, or people connected on social media.

Trees are special graphs that look like family trees - they branch out from a main point without forming loops. Your computer's file folders are organized like a tree.

5.8.3 Game Theory

Game theory studies how people make decisions when what they choose affects others, and what others choose affects them. It's like figuring out the best strategy in a game where everyone is trying to win.

The classic example is two prisoners who have to decide whether to confess or stay quiet, not knowing what the other will do.

5.8.4 How They Connect

These ideas work together in many real-world problems. You might use fuzzy logic to handle uncertainty, graphs to show relationships, and game theory to find the best strategy.